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Partial Hausdorff metric and Nadler's fixed point theorem on partial metric spaces

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ABSTRACT

In this paper, we introduce the concept of a partial Hausdorff metric. We initiate study of fixed point theory for multi-valued mappings on partial metric space using the partial Hausdorff metric and prove an analogous to the well-known Nadler's fixed point theorem. Moreover, we give a homotopy result as application of our main result.

1. Introduction and preliminaries

Let (X,d) be a metric space and $CB(X)$ denotes the collection of all nonempty closed and bounded subsets of X. For $A,B \in CB(X)$, define

$$H(A,B) := \max\Big\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(b,A)\Big\},$$

where $d(x,A) := \inf\{d(x,a)\colon a \in A\}$ is the distance of a point x to the set A. It is known that H is a metric on $CB(X)$, called the Hausdorff metric induced by the metric d.

Definition 1.1. Let X be any nonempty set. An element x in X is said to be a fixed point of a multi-valued mapping $T : X \to 2^X$ if $x \in Tx$, where 2^X denotes the collection of all nonempty subsets of X.

We recall that a multi-valued mapping $T : X \to CB(X)$ is said to be a contraction if

$$H(Tx,Ty) \leqslant kd(x,y)$$

for all $x,y \in X$ and for some k in $[0,1)$.

The study of fixed points for multi-valued contractions using the Hausdorff metric was initiated by Nadler [14] who proved the following theorem.

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Theorem 1.2. *([14]) Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a contraction mapping. Then, there exists $x \in X$ such that $x \in Tx$.*

Later, an interesting and rich fixed point theory was developed. The theory of multi-valued maps has application in control theory, convex optimization, differential equations and economics (see also [6]). On the other hand, Matthews [9] introduced the concept of a partial metric as a part of the study of denotational semantics of dataflow networks. He gave a modified version of the Banach contraction principle, more suitable in this context (see also [7,10]). In fact, (complete) partial metric spaces constitute a suitable framework to model several distinguished examples of the theory of computation and also to model metric spaces via domain theory (see, [5,8,9,11–13]). In this paper, we introduce the concept of a partial Hausdorff metric and extend the Nadler's fixed point theorem on partial metric spaces using the partial Hausdorff metric.

In the sequel the letters $\mathbb{R}$, $\mathbb{R}^+$ and $\mathbb{N}^*$ will denote the set of all real numbers, the set of all nonnegative real numbers and the set of all positive integer numbers, respectively.

Consistent with [2,3,9], the following definitions and results will be needed in the sequel.

Definition 1.3. Let X be a nonempty set. A function $p : X \times X \to \mathbb{R}^+$ is said to be a partial metric on X if for any $x, y, z \in X$, the following conditions hold:

(P_1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$;
(P_2) $p(x,x) \leqslant p(x,y)$;
(P_3) $p(x,y) = p(y,x)$;
(P_4) $p(x,z) \leqslant p(x,y) + p(y,z) - p(y,y)$.

The pair (X, p) is then called a partial metric space.

If $p(x,y) = 0$, then (P_1) and (P_2) imply that $x = y$. But the converse does not hold always.

A trivial example of a partial metric space is the pair $(\mathbb{R}^+, p)$, where $p : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined as $p(x,y) = \max\{x,y\}$ (see also [1]).

Example 1.4. ([9]) If $X = \{[a,b]\colon\ a, b \in \mathbb{R},\ a \leqslant b\}$, then $p([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$ defines a partial metric p on X.

For some more examples of partial metric spaces, we refer to [2,4,11,13].

Each partial metric p on X generates a T_0 topology τ_p on X which has as a base the family open p-balls $\{B_p(x,\varepsilon)\colon\ x \in X, \varepsilon > 0\}$, where $B_p(x,\varepsilon) = \{y \in X\colon\ p(x,y) < p(x,x) + \varepsilon\}$, for all $x \in X$ and $\varepsilon > 0$.

Observe (see [9, p. 187]) that a sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$, with respect to τ_p, if and only if $p(x,x) = \lim_{n\to\infty} p(x,x_n)$.

If p is a partial metric on X, then the function $p^s : X \times X \to \mathbb{R}^+$ given by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$, defines a metric on X.

Furthermore, a sequence $\{x_n\}$ converges in (X, p^s) to a point $x \in X$ if and only if

$$\lim_{n,m\to\infty} p(x_n,x_m) = \lim_{n\to\infty} p(x_n,x) = p(x,x). \tag{1.1}$$

Definition 1.5. ([9]) Let (X,p) be a partial metric space.

(a) A sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{n,m\to\infty} p(x_n,x_m)$ exists and is finite.
(b) (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges with respect to τ_p to a point $x \in X$ such that $\lim_{n\to\infty} p(x,x_n) = p(x,x)$. In this case, we say that the partial metric p is complete.

Lemma 1.6. *([2,9]) Let (X,p) be a partial metric space. Then:*

(a) *A sequence $\{x_n\}$ in X is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in metric space (X,p^s).*
(b) *A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete.*

2. Partial Hausdorff metric

Let (X,p) be a partial metric space. Let $CB^p(X)$ be the family of all nonempty, closed and bounded subsets of the partial metric space (X,p), induced by the partial metric p. Note that Closedness is take from (X,τ_p) (τ_p is the topology induced by p) and boundedness is given as follows: A is a bounded subset in (X,p) if there exist $x_0 \in X$ and $M \geqslant 0$ such that for all $a \in A$, we have $a \in B_p(x_0,M)$, that is, $p(x_0,a) < p(a,a) + M$.

For $A, B \in CB^p(X)$ and $x \in X$, define

$$p(x, A) = \inf\{p(x, a),\ a \in A\}, \qquad \delta_p(A, B) = \sup\{p(a, B):\ a \in A\} \quad \text{and}$$
$$\delta_p(B, A) = \sup\{p(b, A):\ b \in B\}.$$

It is immediate to check that $p(x, A) = 0 \Rightarrow p^s(x, A) = 0$ where $p^s(x, A) = \inf\{p^s(x, a),\ a \in A\}$.

Remark 2.1. ([2]) Let (X, p) be a partial metric space and A any nonempty set in (X, p), then

$$a \in \overline{A} \quad \text{if and only if} \quad p(a, A) = p(a, a), \tag{2.1}$$

where $\overline{A}$ denotes the closure of A with respect to the partial metric p. Note that A is closed in (X, p) if and only if $A = \overline{A}$.

Now, we shall study some properties of mapping $\delta_p : CB^p(X) \times CB^p(X) \to [0, \infty)$.

Proposition 2.2. *Let (X, p) be a partial metric space. For any $A, B, C \in CB^p(X)$, we have the following*:

(i) $\delta_p(A, A) = \sup\{p(a, a):\ a \in A\}$;
(ii) $\delta_p(A, A) \leqslant \delta_p(A, B)$;
(iii) $\delta_p(A, B) = 0$ *implies that* $A \subseteq B$;
(iv) $\delta_p(A, B) \leqslant \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Proof. (i) From (2.1), if $A \in CB^p(X)$, then for all $a \in A$, we have $p(a, A) = p(a, a)$ as $\overline{A} = A$. Therefore $\delta_p(A, A) = \sup\{p(a, A):\ a \in A\} = \sup\{p(a, a):\ a \in A\}$.

(ii) Let $a \in A$. Since $p(a, a) \leqslant p(a, b)$ for all $b \in B$, therefore we have $p(a, a) \leqslant p(a, B) \leqslant \delta_p(A, B)$. From (i), we conclude that $\delta_p(A, A) = \sup\{p(a, a):\ a \in A\} \leqslant \delta_p(A, B)$.

(iii) Suppose that $\delta_p(A, B) = 0$. Consequently $p(a, B) = 0$ for all $a \in A$. From (i) and (ii) it follows that $p(a, a) \leqslant \delta_p(A, B) = 0$ for all $a \in A$. That is, $p(a, a) = 0$ for all $a \in A$, and hence $p(a, B) = p(a, a)$ for all $a \in A$. Using (2.1), we have $a \in \overline{B} = B$ whenever $a \in A$ so $A \subseteq B$.

(iv) Let $a \in A$, $b \in B$ and $c \in C$. As

$$p(a, b) \leqslant p(a, c) + p(c, b) - p(c, c),$$

so we have

$$p(a, B) \leqslant p(a, c) + p(c, B) - p(c, c),$$

and

$$p(a, B) + p(c, c) \leqslant p(a, c) + \delta_p(C, B).$$

Since c is an arbitrary element of C, therefore we have

$$p(a, B) + \inf_{c \in C} p(c, c) \leqslant p(a, C) + \delta_p(C, B).$$

As a is an arbitrary element of A, so

$$\delta_p(A, B) \leqslant \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c). \qquad \square$$

Let (X, p) be a partial metric space. For $A, B \in CB^p(X)$, define

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

Proposition 2.3. *Let (X, p) be a partial metric space. For all $A, B, C \in CB^p(X)$, we have*

(h1) $H_p(A, A) \leqslant H_p(A, B)$;
(h2) $H_p(A, B) = H_p(B, A)$;
(h3) $H_p(A, B) \leqslant H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$.

Proof. From (ii) of Proposition 2.2, we have $H_p(A,A) = \delta_p(A,A) \leqslant \delta_p(A,B) \leqslant H_p(A,B)$. By definition, (h2) holds obviously. Now using property (iv) of Proposition 2.2. We have

$$\begin{aligned} H_p(A,B) &= \max\{\delta_p(A,B), \delta_p(B,A)\} \\ &\leqslant \max\Big\{\delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c), \delta_p(B,C) + \delta_p(C,A) - \inf_{c\in C} p(c,c)\Big\} \\ &= \max\{\delta_p(A,C) + \delta_p(C,B), \delta_p(B,C) + \delta_p(C,A)\} - \inf_{c\in C} p(c,c) \\ &\leqslant \max\{\delta_p(A,C), \delta_p(C,A)\} + \max\{\delta_p(C,B), \delta_p(B,C)\} - \inf_{c\in C} p(c,c) \\ &= H_p(A,C) + H_p(C,B) - \inf_{c\in C} p(c,c). \qquad \square \end{aligned}$$

Corollary 2.4. *Let (X,p) be a partial metric space. For $A,B \in CB^p(X)$ the following holds*

$$H_p(A,B) = 0 \quad \textit{implies that} \quad A = B.$$

Proof. Let $H_p(A,B) = 0$. By definition of H_p, $\delta_p(A,B) = \delta_p(B,A) = 0$. Using (iii) of Proposition 2.2, we obtain that $A \subseteq B$ and $B \subseteq A$. Thus, $A = B$. $\square$

Remark 2.5. The converse of Corollary 2.4 is not true in general as it is clear from the following example.

Example 2.6. Let $X = [0,1]$ be endowed with the partial metric $p : X \times X \to \mathbb{R}^+$ defined by

$$p(x,y) = \max\{x,y\}.$$

From (i) of Proposition 2.2, we have

$$H_p(X,X) = \delta_p(X,X) = \sup\{x\colon 0 \leqslant x \leqslant 1\} = 1 \neq 0.$$

In view of Proposition 2.3 and Corollary 2.4, we call the mapping $H_p : CB^p(X) \times CB^p(X) \to [0,+\infty)$, a partial Hausdorff metric induced by p.

Remark 2.7. It is easy to show that any Hausdorff metric is a partial Hausdorff metric. The converse is not true (see Example 2.6).

3. Fixed point of multi-valued contraction mapping

We start with the following lemma needed to prove our main result.

Lemma 3.1. *Let (X,p) be a partial metric space, $A,B \in CB^p(X)$ and $h > 1$. For any $a \in A$, there exists $b = b(a) \in B$ such that*

$$p(a,b) \leqslant hH_p(A,B). \tag{3.1}$$

Proof. If $A = B$, then from (i) of Proposition 2.2, we have

$$H_p(A,B) = H_p(A,A) = \delta_p(A,A) = \sup_{x\in A} p(x,x).$$

Let $a \in A$. Since $h > 1$, therefore we have

$$p(a,a) \leqslant \sup_{x\in A} p(x,x) = H_p(A,B) \leqslant hH_p(A,B).$$

Consequently $b = a$ satisfies (3.1). If $A \neq B$, suppose that there exists $a \in A$ such that $p(a,b) > hH_p(A,B)$ for all $b \in B$. This implies that $\inf\{p(a,y)\colon\ y \in B\} \geqslant hH_p(A,B)$. That is, $p(a,B) \geqslant hH_p(A,B)$. Note that

$$H_p(A,B) \geqslant \delta_p(A,B) = \sup_{x\in A} p(x,B) \geqslant p(a,B) \geqslant hH_p(A,B).$$

Since $A \neq B$, from Corollary 2.4 we have $H_p(A,B) \neq 0$ and above inequality gives $h \leqslant 1$, a contradiction. $\square$

Now, we state and prove our main result.

Theorem 3.2. *Let (X,p) be a complete partial metric space. If $T : X \to CB^p(X)$ is a multi-valued mapping such that for all $x, y \in X$, we have*

$$H_p(Tx,Ty) \leqslant kp(x,y) \tag{3.2}$$

where $k \in (0,1)$. Then T has a fixed point.

Proof. Let $x_0 \in X$ and $x_1 \in Tx_0$. From Lemma 3.1 with $h = \frac{1}{\sqrt{k}}$, there exists $x_2 \in Tx_1$ such that $p(x_1,x_2) \leqslant \frac{1}{\sqrt{k}} H_p(Tx_0,Tx_1)$. As, $H_p(Tx_0,Tx_1) \leqslant kp(x_0,x_1)$ so $p(x_1,x_2) \leqslant \sqrt{k}p(x_0,x_1)$. For $x_2 \in Tx_1$, there exists $x_3 \in Tx_2$ such that $p(x_2,x_3) \leqslant \frac{1}{\sqrt{k}} H_p(Tx_1,Tx_2) \leqslant \sqrt{k}p(x_1,x_2)$. Continuing this process, we obtain a sequence $\{x_n\}$ in X such that

$$x_{n+1} \in Tx_n \quad \text{and} \quad p(x_{n+1},x_n) \leqslant \sqrt{k}p(x_n,x_{n-1}) \quad \text{for all } n \geqslant 1. \tag{3.3}$$

Now from (3.3) and by the mathematical induction, we obtain

$$p(x_{n+1},x_n) \leqslant (\sqrt{k})^n p(x_0,x_1) \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Using (3.4) and the property (P_4) of a partial metric, for any $m \in \mathbb{N}^*$, we have

$$\begin{aligned} p(x_n,x_{n+m}) &\leqslant p(x_n,x_{n+1}) + p(x_{n+1},x_{n+2}) + \cdots + p(x_{n+m-1},x_{n+m}) \\ &\leqslant (\sqrt{k})^n p(x_0,x_1) + (\sqrt{k})^{n+1} p(x_0,x_1) + \cdots + (\sqrt{k})^{n+m-1} p(x_0,x_1) \\ &= \big((\sqrt{k})^n + (\sqrt{k})^{n+1} + \cdots + (\sqrt{k})^{n+m-1}\big) p(x_0,x_1) \\ &\leqslant \frac{(\sqrt{k})^n}{1-\sqrt{k}} p(x_0,x_1) \to 0 \quad \text{as } n \to +\infty \text{ (since } 0<k<1). \end{aligned}$$

By the definition of p^s, we get for any $m \in \mathbb{N}^*$,

$$p^s(x_n,x_{n+m}) \leqslant 2p(x_n,x_{n+m}) \to 0 \quad \text{as } n \to +\infty. \tag{3.5}$$

This yields that $\{x_n\}$ is a Cauchy sequence in (X,p^s). Since (X,p) is complete, then from Lemma 1.6, (X,p^s) is a complete metric space. Therefore, the sequence $\{x_n\}$ converges to some $x^* \in X$ with respect to the metric p^s, that is, $\lim_{n\to+\infty} p^s(x_n,x^*) = 0$. Again, from (1.1), we have

$$p\big(x^*,x^*\big) = \lim_{n\to+\infty} p\big(x_n,x^*\big) = \lim_{n\to+\infty} p(x_n,x_n) = 0. \tag{3.6}$$

Since $H_p(Tx_n,Tx^*) \leqslant kp(x_n,x^*)$, therefore

$$\lim_{n\to+\infty} H_p\big(Tx_n,Tx^*\big) = 0. \tag{3.7}$$

Now $x_{n+1} \in Tx_n$ gives that

$$p\big(x_{n+1},Tx^*\big) \leqslant \delta_p\big(Tx_n,Tx^*\big) \leqslant H_p\big(Tx_n,Tx^*\big).$$

From (3.7), we get

$$\lim_{n\to+\infty} p\big(x_{n+1},Tx^*\big) = 0. \tag{3.8}$$

On the other hand, we have

$$p\big(x^*,Tx^*\big) \leqslant p\big(x^*,x_{n+1}\big) + p\big(x_{n+1},Tx^*\big).$$

Taking limit as $n \to +\infty$ and using (3.6) and (3.8), we obtain $p(x^*,Tx^*) = 0$. Therefore, from (3.6) $(p(x^*,x^*) = 0)$, we obtain

$$p\big(x^*,x^*\big) = p\big(x^*,Tx^*\big),$$

which from (2.1) implies that $x^* \in \overline{Tx^*} = Tx^*$. $\square$

To underline the usefulness of partial metric, we give the following very simple illustrative examples.

Example 3.3. Let $X=\{0,1,4\}$ be endowed with the partial metric $p: X\times X\to\mathbb{R}^+$ defined by

$$p(x,y)=\frac{1}{4}|x-y|+\frac{1}{2}\max\{x,y\}\quad\text{for all } x,y\in X.$$

Note that $p(1,1)=\frac{1}{2}\neq 0$ and $p(4,4)=2\neq 0$, so p is not a metric on X. As $p^s(x,y)=|x-y|$ so (X,p) is a complete partial metric space.

Note that $\{0\}$ and $\{0,1\}$ are bounded sets in (X,p). In fact, if $x\in\{0,1,4\}$, then

$$\begin{aligned} x\in\overline{\{0\}}\quad &\Leftrightarrow\quad p\big(x,\{0\}\big)=p(x,x)\\ &\Leftrightarrow\quad \frac{3}{4}x=\frac{1}{2}x\quad\Leftrightarrow\quad x=0\\ &\Leftrightarrow\quad x\in\{0\}. \end{aligned}$$

Hence $\{0\}$ is closed with respect to the partial metric p. Also

$$\begin{aligned} x\in\overline{\{0,1\}}\quad &\Leftrightarrow\quad p\big(x,\{0,1\}\big)=p(x,x)\\ &\Leftrightarrow\quad \min\left\{\frac{3}{4}x,\frac{1}{4}|x-1|+\frac{1}{2}\max\{x,1\}\right\}=\frac{1}{2}x\\ &\Leftrightarrow\quad x\in\{0,1\}. \end{aligned}$$

Hence $\{0,1\}$ is closed with respect to the partial metric p.

Now, define the mapping $T: X\to CB^p(X)$ by

$$T(0)=T(1)=\{0\}\quad\text{and}\quad T(4)=\{0,1\}.$$

We shall show that, for all $x,y\in X$, the contractive condition (3.2) is satisfied with $k=\frac{1}{2}$. For this, we consider the following cases:

- $x,y\in\{0,1\}$. We have

$$H_p\big(T(x),T(y)\big)=H_p\big(\{0\},\{0\}\big)=0,$$

  and (3.2) is satisfied obviously.
- $x\in\{0,1\}$, $y=4$. We have

$$\begin{aligned} H_p\big(T(0),T(4)\big)&=H_p\big(T(1),T(4)\big)\\ &=H_p\big(\{0\},\{0,1\}\big)\\ &=\max\big\{p\big(0,\{0,1\}\big),\max\big\{p(0,0),p(1,0)\big\}\big\}\\ &=\frac{3}{4}\leqslant\frac{11}{8}=kp(1,4)<\frac{3}{2}=kp(0,4). \end{aligned}$$

- $x=y=4$. We have

$$\begin{aligned} H_p\big(T(4),T(4)\big)&=H_p\big(\{0,1\},\{0,1\}\big)\\ &=\sup\big\{p(x,x)\colon x\in\{0,1\}\big\}\\ &=\max\big\{p(0,0),p(1,1)\big\}\\ &=\frac{1}{2}\leqslant 1=kp(4,4). \end{aligned}$$

Thus, all the hypotheses of Theorem 3.2 are satisfied. Here, $x=0$ is a fixed point of T.

Example 3.4. Let $X=\{0,1,2\}$ be endowed with the partial metric $p: X\times X\to\mathbb{R}^+$ defined by

$$\begin{aligned} &p(0,0)=p(1,1)=0,\qquad p(2,2)=\frac{1}{4},\\ &p(0,1)=p(1,0)=\frac{1}{3},\\ &p(0,2)=p(2,0)=\frac{11}{24},\\ &p(1,2)=p(2,1)=\frac{1}{2}. \end{aligned}$$

Define the mapping $T : X \to CB^p(X)$ by

$$T(0) = T(1) = \{0\} \quad \text{and} \quad T(2) = \{0, 1\}.$$

Note that, Tx is closed and bounded for all $x \in X$ under the given partial metric space (X, p). We shall show that, for all $x, y \in X$, the contractive condition (3.2) is satisfied with $k = \frac{3}{4}$. For this, we consider the following cases:

- $x, y \in \{0, 1\}$. We have

$$H_p\big(T(x), T(y)\big) = H_p\big(\{0\}, \{0\}\big) = 0,$$

 and (3.2) is satisfied obviously.
- $x \in \{0, 1\}$, $y = 2$. We have

$$\begin{aligned} H_p\big(T(1), T(2)\big) &= H_p\big(T(0), T(2)\big) \\ &= H_p\big(\{0\}, \{0, 1\}\big) \\ &= \max\big\{p\big(0, \{0, 1\}\big), \max\big\{p(0, 0), (1, 0)\big\}\big\} \\ &= \frac{1}{3} \leqslant \frac{11}{24}k = kp(0, 2) < \frac{1}{2}k = kp(1, 2). \end{aligned}$$

- $x = y = 2$. We have

$$\begin{aligned} H_p\big(T(2), T(2)\big) &= H_p\big(\{0, 1\}, \{0, 1\}\big) \\ &= \sup\big\{p(x, x) \colon x \in \{0, 1\}\big\} \\ &= \max\big\{p(0, 0), p(1, 1)\big\} \\ &= 0 < \frac{1}{4}k = kp(2, 2). \end{aligned}$$

Thus, all the conditions of Theorem 3.2 are satisfied. Here, $x = 0$ is a fixed point of T.

On the other hand, the metric p^s induced by the partial metric p is given by

$$\begin{aligned} &p^s(0, 0) = p^s(1, 1) = p^s(2, 2) = 0, \\ &p^s(0, 1) = p^s(1, 0) = p^s(0, 2) = p^s(2, 0) = \frac{2}{3}, \\ &p^s(2, 1) = p^s(1, 2) = \frac{3}{4}. \end{aligned}$$

Now, it is easy to show that Theorem 1.2 is not applicable in this case. Indeed, for $x = 0$ and $y = 2$, we have

$$\begin{aligned} H\big(T(0), T(2)\big) &= H\big(\{0\}, \{0, 1\}\big) \\ &= \max\big\{\sup\big\{p^s\big(0, \{0, 1\}\big)\big\}, \sup\big\{p^s\big(\{0, 1\}, \{0\}\big)\big\}\big\} \\ &= \max\left\{0, \frac{2}{3}\right\} = \frac{2}{3} \nleqslant \frac{2}{3}k = kp^s(0, 2), \end{aligned}$$

for any $k \in (0, 1)$.

4. An application

In this section, as application of our main result, we derive a homotopy result.

Theorem 4.1. *Let (X, p) be a complete partial metric space, A be an open subset of X and C be a closed subset of X, with $A \subset C$. Let $F : C \times [0, 1] \to CB^p(X)$ be an operator such that the following conditions are satisfied*:

(a) *$x \notin F(x, t)$ for each $x \in C \backslash A$ and each $t \in [0, 1]$,*
(b) *there exists $k \in (0, 1)$ such that for each $t \in [0, 1]$ and each $x, y \in C$ we have*

$$H_p\big(F(x, t), F(y, t)\big) \leqslant kp(x, y),$$

(c) *there exists a continuous function* $\eta : [0,1] \to \mathbb{R}$ *such that*

$$H_p\big(F(x,t), F(x,s)\big) \leqslant k\big|\eta(t) - \eta(s)\big| \quad \text{for all } t, s \in [0,1] \text{ and each } x \in C,$$

(d) *if* $x \in F(x,t)$ *then* $F(x,t) = \{x\}$.

Then $F(\cdot, 0)$ *has a fixed point if and only if* $F(\cdot, 1)$ *has a fixed point.*

Proof. Consider the set

$$Q := \big\{t \in [0,1] \,\big|\, x \in F(x,t) \text{ for some } x \in A\big\}.$$

Since $F(\cdot, 0)$ has a fixed point and (a) holds, we have that $0 \in Q$, so Q is a nonempty set. We will show that Q is both closed and open in $[0,1]$, and so by the connectedness of $[0,1]$ we are finished since $Q = [0,1]$.

First, let us prove that Q is open in $[0,1]$. Let $t_0 \in Q$ and $x_0 \in A$ with $x_0 \in F(x_0, t_0)$. Since A is open in (X, p), there exists $r > 0$ such that $B_p(x_0, r) \subseteq A$. Consider $\varepsilon = r + p(x_0, x_0) - k(r + p(x_0, x_0)) > 0$. Since η is continuous on t_0, there exists $\alpha(\varepsilon) > 0$ such that $|\eta(t) - \eta(t_0)| < \varepsilon$ for all $t \in (t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon))$.

Let $t \in (t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon))$, for $x \in \overline{B_p(x_0, r)} = \{x \in X \mid p(x_0, x) \leqslant p(x_0, x_0) + r\}$, we have

$$\begin{aligned} p\big(F(x,t), x_0\big) &\leqslant H_p\big(F(x,t), F(x_0, t_0)\big) \\ &\leqslant H_p\big(F(x,t), F(x, t_0)\big) + H_p\big(F(x, t_0), F(x_0, t_0)\big) \\ &\leqslant k\big|\eta(t) - \eta(t_0)\big| + kp(x, x_0) \\ &\leqslant k\varepsilon + k\big(p(x_0, x_0) + r\big) \\ &= k\big[r + p(x_0, x_0) - k\big(p(x_0, x_0) + r\big)\big] + k\big(p(x_0, x_0) + r\big) \\ &< r + p(x_0, x_0) - k\big(p(x_0, x_0) + r\big) + k\big(p(x_0, x_0) + r\big) \\ &= r + p(x_0, x_0). \end{aligned}$$

Then, for each fixed $t \in (t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon))$, $F(\cdot, t) : \overline{B_p(x_0, r)} \to CB^p(X)$ satisfies all the hypotheses of Theorem 3.2 and so $F(\cdot, t)$ has a fixed point in $\overline{B_p(x_0, r)} \subset C$. But this fixed point must be in A since (a) holds. Hence $(t_0 - \alpha(\varepsilon), t_0 + \alpha(\varepsilon)) \subseteq Q$ and therefore Q is open in $[0,1]$.

Next, we show that Q is closed in $[0,1]$. To see this, let $\{t_n\}_{n \in \mathbb{N}^*}$ be a sequence in Q with $t_n \to t^* \in [0,1]$ as $n \to +\infty$. We must show that $t^* \in Q$. By the definition of Q, for all $n \in \mathbb{N}^*$, there exists $x_n \in A$ with $x_n \in F(x_n, t_n)$. Then, for $m, n \in \mathbb{N}^*$, we have using (d) and (h3)

$$\begin{aligned} p(x_n, x_m) &= H_p\big(F(x_n, t_n), F(x_m, t_m)\big) \\ &\leqslant H_p\big(F(x_n, t_n), F(x_n, t_m)\big) + H_p\big(F(x_n, t_m), F(x_m, t_m)\big) \\ &\leqslant k\big|\eta(t_n) - \eta(t_m)\big| + kp(x_n, x_m). \end{aligned}$$

This implies that

$$p(x_n, x_m) - kp(x_n, x_m) \leqslant k\big|\eta(t_n) - \eta(t_m)\big|,$$

and we get

$$p(x_n, x_m) \leqslant \frac{k}{1-k}\big|\eta(t_n) - \eta(t_m)\big|.$$

Since η is continuous and $\{t_n\}_{n \in \mathbb{N}^*}$ is convergent, letting $n, m \to +\infty$ in the above inequality, we obtain $\lim_{n,m \to +\infty} p(x_n, x_m) = 0$, that is, $\{x_n\}_{n \in \mathbb{N}^*}$ is a Cauchy sequence in (X, p). Since (X, p) is complete, there exists $x^* \in C$ with $p(x^*, x^*) = \lim_{n \to +\infty} p(x^*, x_n) = \lim_{n,m \to +\infty} p(x_n, x_m) = 0$.

On the other hand, we have

$$\begin{aligned} p\big(x_n, F\big(x^*, t^*\big)\big) &\leqslant H_p\big(F(x_n, t_n), F\big(x^*, t^*\big)\big) \\ &\leqslant H_p\big(F(x_n, t_n), F\big(x_n, t^*\big)\big) + H_p\big(F\big(x_n, t^*\big), F\big(x^*, t^*\big)\big) \\ &\leqslant k\big|\eta(t_n) - \eta\big(t^*\big)\big| + kp\big(x_n, x^*\big). \end{aligned}$$

Letting $n \to +\infty$ in the above inequality, we get $\lim_{n \to +\infty} p(x_n, F(x^*, t^*)) = 0$ and so

$$p\big(x^*, F\big(x^*, t^*\big)\big) = \lim_{n \to +\infty} p\big(x_n, F\big(x^*, t^*\big)\big) = 0,$$

which implies that $x^* \in F(x^*, t^*)$, and since (a) holds, we have $x^* \in A$. Thus $t^* \in Q$ and Q is closed in $[0,1]$.

One can use the same strategy to prove the reverse implication. This completes the proof. $\square$

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